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2007 J. Phys. A: Math. Theor. 40 8069

(http://iopscience.iop.org/1751-8121/40/28/S15)

J. Phys. A: Math. Theor. **40** (2007) 8069–8080 doi:10.1088/1751-8113/40/28/S15

Qubits in a random environment

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Received 21 December 2006, in final form 19 March 2007
Published 27 June 2007
Online at stacks.iop.org/JPhysA/40/8069

Abstract
Decoherence phenomena in a small quantum system coupled to a complex environment can be modelled with random matrices. We propose a simple deterministic model in the limit of a high dimensional environment. The model is investigated numerically and some analytically addressable questions are singled out.

PACS numbers: 03.65.Yz, 03.67.La

1. Introduction

Systems like single ultra-cold atoms, well-isolated from the outside world, are often modelled by the spin-boson model [1]. Due to the low temperature only the ground state and the first few excited states are relevant, the atom is therefore essentially a spin. Even if the atom is trapped in a region of high vacuum there is still the unavoidable interaction with the surrounding radiation field, this is the boson part of the model. Moreover, the radiation field is assumed to be thermal at a low temperature.

A different type of situation arises when our distinguished quantum system is no longer trapped in a region of high vacuum but is embedded within a solid [2]. If the system only weakly interacts with neighbouring atoms or molecules, we might imagine that its environment consists of a multi-level system of rather unknown structure interacting weakly with our distinguished spin. The precise interaction is difficult to know and details may vary from sample to sample. Following the approach of Wigner with the spectral lines of large nuclei [3], we arrive at a model of a spin randomly coupled to a large system the proper dynamics of which is also random.

The simplest example of such a model is that of a qubit coupled to a random environment. Both the coupling and the own dynamics of the environment are drawn from the Gaussian unitary ensemble (GUE). A further simplification is to make the environment infinite. This can be done in a mathematically consistent way by properly scaling the variances of the

random matrices with their dimension. The resulting structure is quite appealing and goes under the name of free probability [4, 5]. We briefly present a simple model in this vein but postpone its analysis to a forthcoming paper. The aim of this paper is to investigate the model by numerical simulations and to single out in this way a number of questions that can be addressed analytically.

The Hamiltonian governing the dynamics of the coupled system, qubit and environment, has the following form:

$$h = \begin{pmatrix} \varepsilon + x & \lambda y \\ \lambda y & x \end{pmatrix}. \tag{1}$$

The gap between the ground and excited state of the qubit is ε and λ is the strength of the coupling with the environment. Furthermore, x and y are Hermitian environment operators describing the proper dynamics of the environment and the nature of its coupling with the qubit.

Initially, the qubit is assumed to be completely decoupled from the environment and described by a density matrix

$$\rho_0 = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

with

$$\rho_{11} \geqslant 0, \qquad \rho_{11} + \rho_{22} = 1, \qquad \rho_{21} = \overline{\rho_{12}} \qquad \text{and} \qquad |\rho_{12}|^2 \leqslant \rho_{11}\rho_{22}.$$

Equivalently

$$\rho_0 = \tfrac{1}{2}(\mathbf{1} + \overline{r}_0 \cdot \overline{\sigma})$$

where $\overline{\sigma}$ are the Pauli matrices and $\overline{r}_0$ is the three-dimensional Bloch vector corresponding to ρ_0

$$\overline{r}_0 = (2\,\mathrm{Re}(\rho_{12}), -2\,\mathrm{Im}(\rho_{12}), 2\rho_{11} - 1).$$

The reduced dynamics of the qubit is given by

$$t \mapsto \langle \mathrm{tr}(\rho_0 u(t) \overline{\sigma} u(t)^\dagger) \rangle. \tag{2}$$

In this formula $u(t)$ is the global Hamiltonian dynamics $\exp(-\mathrm{i}th)$, we put $\hbar = 1$, and $\langle\,\rangle$ is the average over the environment.

It remains to specify the nature of the environment observables x and y. Unlike the classical case, where $[x, y] = 0$, or the case of position and momentum fields, where $[x, y] = -\mathrm{i}$, there is no commutation relation between x and y which allows us to simplify monomials. The only rules that can be applied are of the type $x^m x^n = x^{m+n}$. The term 'free' coins this lack of simplification rules.

In order to compute the expectation of an arbitrary polynomial in x and y we need to know the mixed moments of x and y

$$\langle x^{m_1} y^{n_1} x^{m_2} \cdots y^{n_k} \rangle.$$

The second ingredient in the structure of the environment is independence which determines the expectations of arbitrary monomials in x and y in terms of expectations of powers of x and y alone. This is expressed by saying that x and y are in free relation. It amounts to impose that

$$\langle (x^{m_1} - \langle x^{m_1} \rangle)(y^{n_1} - \langle y^{n_1} \rangle) \cdots (y^{n_k} - \langle y^{n_k} \rangle) \rangle = 0 \tag{3}$$

which allows us to recursively compute all mixed moments in terms of the moments of x and y. For example,

$$\langle xyxy\rangle = \langle x^2\rangle\langle y\rangle^2 + \langle x\rangle^2\langle y^2\rangle - \langle x\rangle^2\langle y\rangle^2.$$

A further natural choice is to take x and y standard semi-circularly distributed, this is the analogue of the standard normal distribution in usual probability theory

$$\langle x^{2n+1}\rangle = \langle y^{2n+1}\rangle = 0 \qquad \text{and} \qquad \langle x^{2n}\rangle = \langle y^{2n}\rangle = \frac{\binom{2n}{n}}{n+1} = \mathrm{c}_n, \tag{4}$$

where c_n's are the Catalan numbers.

An important result in free probability, which has been vastly extended, is that freeness with the standard semi-circular distribution of (4) arises in the limit of large random matrices [4, 5]. Wigner showed that the eigenvalue distribution of Gaussian random matrices of dimension d asymptotically tends to a standard semi-circle (4) when the variances of the independent and identically distributed matrix entries are chosen as $1/\sqrt{d}$ (real Gaussian random variables on the diagonal and complex ones in the upper triangle). The mixed moments of n-tuples of such random matrices, computed with respect to the normalized trace, have been shown to satisfy the freeness relation (3) in the limit $d \to \infty$. It is our aim here to numerically analyse the reduced dynamics (2) in terms of the dimension of the environment and the control parameters ε and λ and to single out a number of questions that could be addressed analytically.

2. The basic model

In order to distinguish between the abstract freely independent elements $x, y, \ldots$ of the introduction which are semi-circularly distributed and their finite-dimensional random approximants, we shall use upper case symbols $X, Y, \ldots$ to denote the latter.

Drawing random Hermitian matrices X and Y from the d-dimensional GUE with unit variance, we obtain a random Hamiltonian

$$H = \begin{pmatrix} \varepsilon + X & \lambda Y \\ \lambda Y & X \end{pmatrix}$$

corresponding to (1).

Instead of considering the reduced dynamics (2), we shall use the interaction picture and consider

$$t \mapsto \left\langle \mathrm{tr}\left(\rho_0 U_0^\dagger(t) U(t)\overline{\sigma} U(t)^\dagger U_0(t)\right)\right\rangle =: \mathrm{tr}\,\rho(t)\overline{\sigma}. \tag{5}$$

where $U_0(t) := \exp(-\mathrm{i}tH_0)$ is the free dynamics of the uncoupled system

$$H_0 := \begin{pmatrix} \varepsilon + X & 0 \\ 0 & X \end{pmatrix}.$$

The full dynamics $U(t)$ is $\exp(-\mathrm{i}tH)$. As both H and H_0 have discrete eigenvalues, the expectation values (5) are almost periodic in time and one cannot expect some definite limiting behaviour. The revival time depends on the details of the spectrum of the Hamiltonian which consists of a regular part with some superimposed fluctuations. The analysis of the energy level separation as a function of the system size is one of the central questions quantum chaos is concerned with and will not be considered in this paper, see however [6, 7]. In our case, the revival time appears however to increase drastically with the dimension.

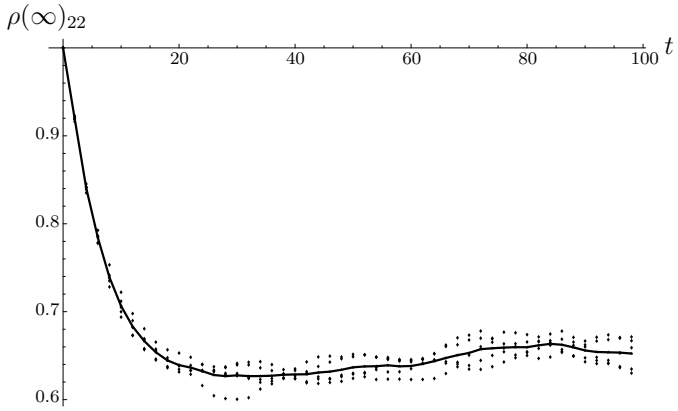

Figure 1. Five realizations of the reduced density.

In [8], the behaviour of the reduced dynamics was considered from a different point of view. The normalized trace over the environment was replaced by the micro canonical ensemble determined by the free Hamiltonian of the environment and the average over the GUE was considered with respect to Y. Several scaling limits were considered.

2.1. Deterministic dynamics

A number of observations emerge from our numerical evidence. The reduced density matrices $\rho(t)$ clearly depend on the initial condition ρ_0, on the parameters ε and λ and also on the d-dimensional matrices X and Y. A striking observation is that for sufficiently large dimensions and keeping all other parameters fixed, the reduced density matrix becomes almost independent of the specific realization of the random X and Y.

In figure 1, we plotted the evolution up to time 100 of the (22) entry of the reduced density for five realizations of the random matrices X and Y. The dimension of the environment is 30, $\varepsilon = 1$ and $\lambda = 0.2$. The solid line is the average. Obviously, for such a low dimension of the environment, the evolution of the reduced density matrix still strongly depends on the realization of the random X and Y.

In figure 2, the averaged variance of five realizations of the (22) entry of the reduced density matrix with respect to their average over 100 time steps is plotted as a function of the dimension. Initially the qubit is in its ground state, $\varepsilon = 1$ and $\lambda = 0.2$. In figure 3, the estimated long time limit over five realizations is plotted versus the dimension of the environment. Finally, in figure 4 the variance of the estimated limit versus dimension is plotted.

From now on we shall consider a sufficiently high dimension of the environment so that we can assume that a typical realization of the randomness generates the probability one behaviour of the limit $d \to \infty$. As free probability is the limit of large random matrices the deterministic behaviour of the reduced density matrix is the model with free independent observables x and y that was sketched in the introduction. Convincing evidence for this is given by the perturbative computation of the reduced density matrix in the free model. To achieve this we use the Dyson expansion in the interaction picture. Decomposing the Hamiltonian into a free and a perturbed part

$$h = h_0 + \lambda p$$

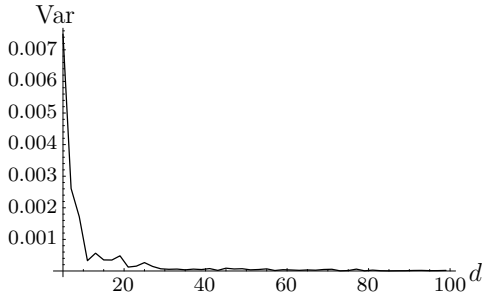

Figure 2. Averaged variance as a function of the dimension.

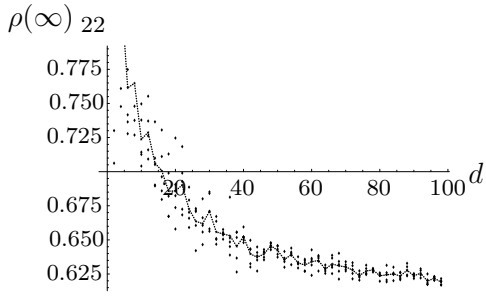

Figure 3. Long time limit as a function of the dimension.

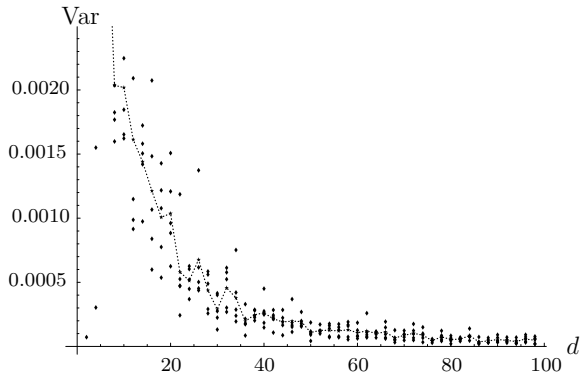

Figure 4. Variance of the estimated limit as a function of the dimension.

we have for an observable a of the total system

$$u_0^\dagger(t)u(t)au^\dagger(t)u_0(t) = a + (\mathrm{i}\lambda)\int_0^t \mathrm{d}s_1\big[p_{t-s_1}, a\big] + \cdots$$
$$+ (\mathrm{i}\lambda)^n \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n \big[p_{t-s_1}, \big[p_{s_1-s_2}, \ldots \big[p_{s_{n-1}-s_n}, a\big]\ldots\big]\big] + \cdots.$$

Here p_t is the freely evolved perturbation at time t.

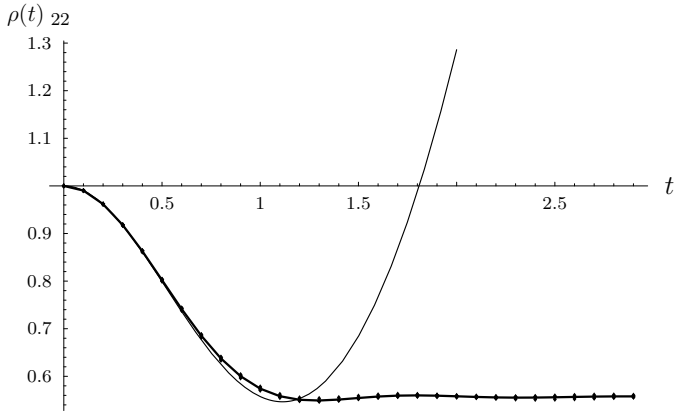

Figure 5. Dyson expansion (thin line) versus simulated evolution (thick line).

We now apply the Dyson expansion to the computation of the reduced density matrix. Using the rules (3) of freeness and the semicircular distribution of x and y, we can expand the reduced density matrix in orders of λ. Because of symmetry reasons only terms of even order will contribute. The final expression can be written in terms of integrals of moments of y and of expectations of $\exp(\mathrm{i}sx)$. This last expression is the characteristic function of a standard semi-circular random variable

$$\langle \exp(\mathrm{i}sx)\rangle = \frac{1}{2\pi}\int_{-2}^{2} \mathrm{d}\zeta \sqrt{4-\zeta^2}\mathrm{e}^{\mathrm{i}s\zeta} = \frac{\mathrm{J}_1(2s)}{s}.$$

Setting the initial state equal to the ground state of the qubit, a rather tedious computation leads to the following expectation for $(1-\sigma^z)/2$, i.e. the (22) component of the reduced density matrix,

$$\begin{aligned}1 - 2\lambda^2 \int_0^t \mathrm{d}x\,(t-x)\frac{\cos(\varepsilon x)\mathrm{J}_1^2(2x)}{x^2} + \frac{4}{3}\lambda^4 \int_0^t \mathrm{d}x_1 \int_0^t \mathrm{d}x_2 (t-x_1)(t-x_2) \\ \times \left\{ \cos(\varepsilon(x_1-x_2))\frac{\mathrm{J}_1^2(2x_1)\mathrm{J}_1^2(2x_2)\mathrm{J}_1^2(2(x_1-x_2))}{x_1^2x_2^2(x_1-x_2)^2} + \cos(\varepsilon(x_1+x_2)) \right. \\ \left. \times \frac{\mathrm{J}_1^2(2x_1)\mathrm{J}_1^2(2x_2)\mathrm{J}_1^2(2(x_1+x_2))}{x_1^2x_2^2(x_1+x_2)^2} \right\} + O(\lambda^6).\end{aligned}$$

The next order is still analytically computable, the obtained expression is however rather involved.

In figure 5, we plot the evolution of the (22) entry according to the Dyson expansion truncated at order 6 versus an evolution obtained by numerical simulation considering an environment of dimension 100. The parameters are $\varepsilon = 1$ and $\lambda = 1$.

2.2. The asymptotic regime

Another observation that emerges from the numerics is that the reduced density matrix rapidly seems to tend to a limit when t becomes large. Of course, due to the finite dimensions, there is a revival time and there is certainly no limiting behaviour with respect to time. However if we first take the limit $d \to \infty$ we can certainly expect a definite long-term behaviour of the basic deterministic model introduced in (1), i.e. the infinite random environment drives the qubit to

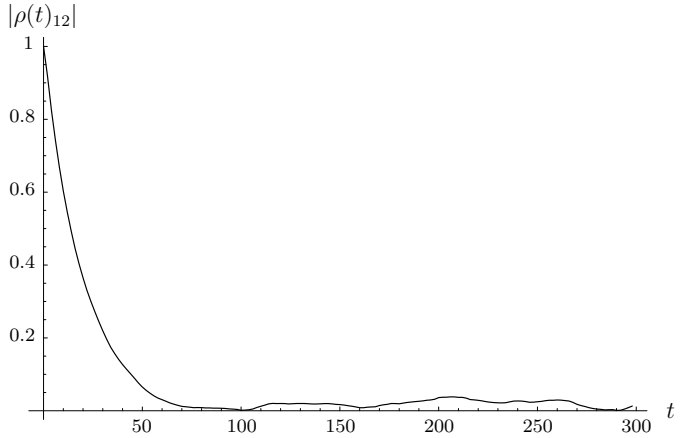

Figure 6. Evolution of off-diagonal entry.

a final state ρ_∞ which obviously depends on the initial condition ρ_0 and on the parameters ε and λ.

Let us choose $\varepsilon > 0$ and compute the reduced functional obtained from an initial condition

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{12} = \operatorname{tr} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

This initial condition is not a qubit expectation but we can formally compute its reduced evolution. It turns out that it converges to zero. In figure 6, the absolute value of the off-diagonal matrix entry of the reduced functional is shown for a single realization in dimension 100, $\varepsilon = 1$ and $\lambda = 0.2$.

Let us now assume that initially the qubit is in a state possibly mixed but reasonably close to the ground state. We decompose this state as

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = (1 - 2\rho_{11}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 2\rho_{11} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \rho_{12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \rho_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By linearity, the reduced density matrix will decompose in the same way. In the limit $t \to \infty$, the contributions of the last two terms vanish as discussed above. Moreover, the second term is constant in time. We restrict therefore from now on our attention to the ground state of the qubit as initial condition.

It is clear from the simulations that the asymptotic reduced state is diagonal in the σ^z eigenbasis: it is a mixture of the ground and the excited state of the qubit. Also the dependence of the limit on the parameters ε and λ is easily understood from an intuitive point of view. A large ε means a large energy gap between the ground and excited state of the qubit. It is rather unlikely that the environment will induce frequent jumps to the excited state and therefore we may expect a rather high proportion of ground state in the long run. It appears in fact from the simulations that

$$\lim_{\varepsilon \to \infty} \rho(\infty)_{22} = 1.$$

Another, rather unphysical, regime is attained when λ is quite large. In this case, the environment rapidly decoheres the qubit and for symmetry reasons one expects to obtain

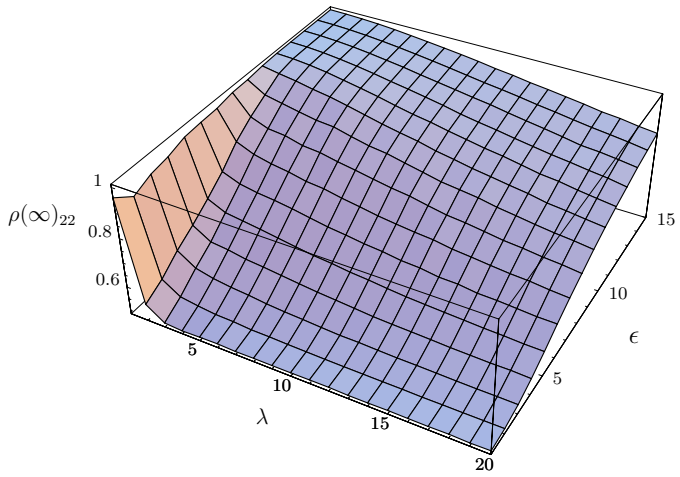

Figure 7. The limiting state as a function of ε and λ.
(This figure is in colour only in the electronic version)

an equal mix of ground and excited state, i.e. the normalized trace on the qubit. Indeed,

$$\lim_{\lambda\to\infty} \rho(\infty)_{22} = \tfrac{1}{2}.$$

In figure 7, we plot the asymptotic value of the (22) entry of the reduced density matrix. The initial state of the qubit is set to the ground state.

These two behaviours point at a discontinuity of the limiting density when both ε and λ are small. In fact, an interesting behaviour seems to emerge when both ε and λ tend to zero in such a way that ε/λ tends to some value $\gamma \geqslant 0$.

Another limiting situation is obtained for both ε and λ large. In this limit, x becomes negligible and we obtain a model of the type

$$h = \begin{pmatrix} \varepsilon & y \\ y & 0 \end{pmatrix}.$$

This model is essentially classical and can be solved explicitly. The obtained expressions are rather complicated but simplify when $t \to \infty$ to

$$\rho(\infty) = \frac{1}{16}\begin{pmatrix} 8+\varepsilon^2-\varepsilon\sqrt{16+\varepsilon^2} & 0 \\ 0 & 8-\varepsilon^2+\varepsilon\sqrt{16+\varepsilon^2} \end{pmatrix}. \tag{6}$$

As before, the initial state is the ground state of the qubit.

In figure 8, we show the estimated asymptotic (22) entry of the reduced density matrix for a $d = 100$ versus the gap of the qubit. Initially, the qubit is in its ground state and the continuous line is the analytical expression (6).

The case $\varepsilon = 0$ and $\lambda > 0$ corresponds to yet another model that can be solved analytically. In this case, we should use the σ^x eigenbasis $e_\pm := \frac{1}{\sqrt{2}}(e_1 \pm e_2)$. The total Hamiltonian then decomposes into a block diagonal matrix with blocks $h_\pm := x \pm \lambda y$. It is possible to compute

$$t \mapsto \langle \exp(\mathrm{i}th_+)\exp(-\mathrm{i}th_-)\rangle \tag{7}$$

and in particular to show that

$$\lim_{t\to\infty} \langle \mathrm{e}^{\mathrm{i}th_+}\mathrm{e}^{-\mathrm{i}th_-}\rangle = 0.$$

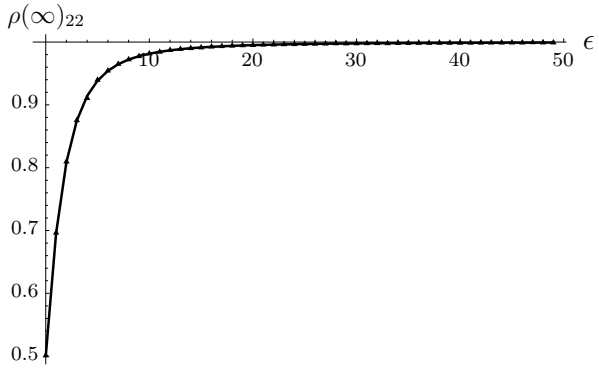

Figure 8. Asymptotic (22) entry versus ϵ, triangles obtained numerically, full line is theoretical expectation.

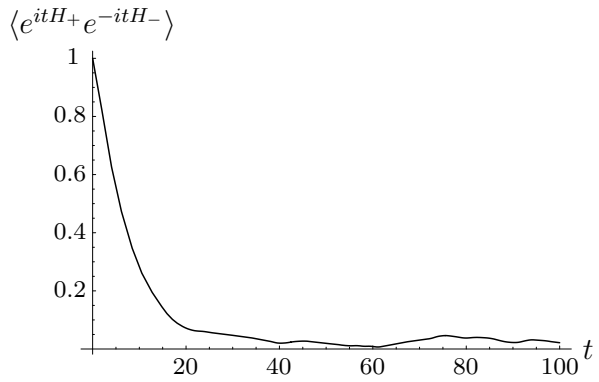

Figure 9. The expectation of $\exp(\mathrm{i}tH_+)\exp(-\mathrm{i}tH_-)$.

When $t \to \infty$, an initial density matrix

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \text{ evolves into } \begin{pmatrix} \frac{1}{2} & \frac{\rho_{12}+\rho_{21}}{2} \\ \frac{\rho_{12}+\rho_{21}}{2} & \frac{1}{2} \end{pmatrix}, \tag{8}$$

which is no longer diagonal in the σ^z eigenbasis.

In figure 9, the absolute value of the random approximation of (7) is plotted for $\lambda = 0.2$, while the small triangles in figure 10 show the real part of the off-diagonal matrix element of the asymptotic state as a function of the initial condition. The continuous line is the analytic expression (8).

3. Extended models

Up to now, we restricted our attention to the basic model introduced in (1) and further analysed in section 2. Obviously, several relevant extensions can and should be considered. The simple interaction in (1) satisfies a reality condition that cannot be expected to hold in general. A

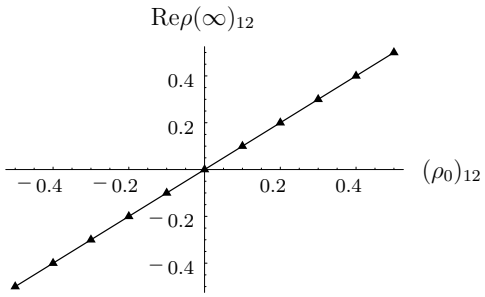

Figure 10. The off-diagonal part of the asymptotic density matrix.

more general Hamiltonian of the form

$$h = \begin{pmatrix} \varepsilon + x & \lambda y + \mu i z \\ \lambda y - \mu i z & X \end{pmatrix}$$

with x, y and z in free relation should be considered.

A more relevant situation from the point of view of quantum information arises when the small system is composite, e.g. two qubits. Unlike the single qubit model, the notion of entanglement is now viable and we are interested in investigating its dynamical behaviour. We may expect that any entanglement should disappear as the qubits interact with the random environment. Judging from the single qubit behaviour that we saw earlier this loss of entanglement would occur because the random environment selects a preferred final mixed state with the off-diagonal elements going to zero.

Several basic models can be envisaged. If the two qubits are sufficiently close to one another they will interact in the same way with the environment and this can be modelled by an interaction term in the Hamiltonian of the form

$$\lambda\left(\sigma_1^x + \sigma_2^x\right) y. \tag{9}$$

Here λ is a common coupling constant and σ_1 and σ_2 denote the Pauli matrices of the first and the second qubit. If the qubits are not that close we should rather consider an interaction term of the form

$$\lambda_1 \sigma_1^x y_1 + \lambda_2 \sigma_2^x y_2. \tag{10}$$

In this expression λ_1 and λ_2 can be quite different and y_1 and y_2 are freely independent.

Establishing manageable quantitative criteria for the degree of entanglement of general mixed many party states is still an open problem. Extending the simple and explicit characterization for separability of a mixed two-qubit state that has been obtained by Wootters [9] seems to offer a promising approach [10]. This kind of criterion goes under the name of concurrence. The expression of the concurrence for a two-qubit state ρ is

$$C(\rho) := \max\{0, r_1 - r_2 - r_3 - r_4\}.$$

Here r_j are the eigenvalues in decreasing order of the matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$. The density matrix $\tilde{\rho}$ is obtained by multiplying the complex conjugated matrix ρ in the standard tensor basis on both sides with $\sigma^y \otimes \sigma^y$. $C(\rho)$ ranges from 0 for no entanglement to 1 for a maximally entangled state.

Let us first display the concurrence in the case of the second model (10). Figure 11 shows two plots of the concurrence for the case of different interactions. Both plots apply to all of the Bell states which are initially maximally entangled. Here we see the behaviour is in

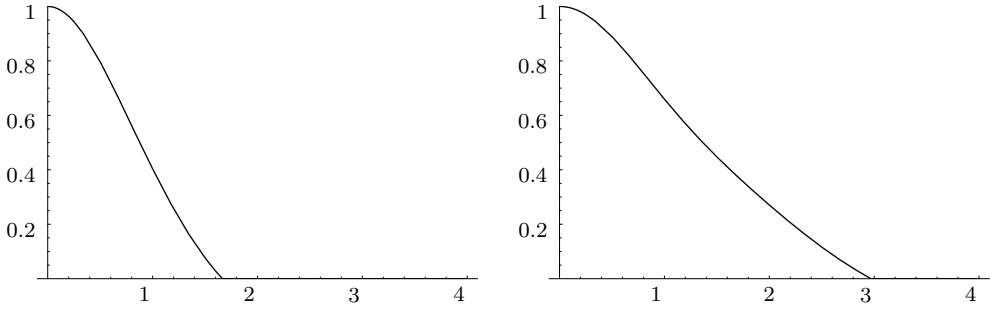

Figure 11. Concurrence versus time with different environment interactions for both qubits (first: same λ, second: has one interaction off).

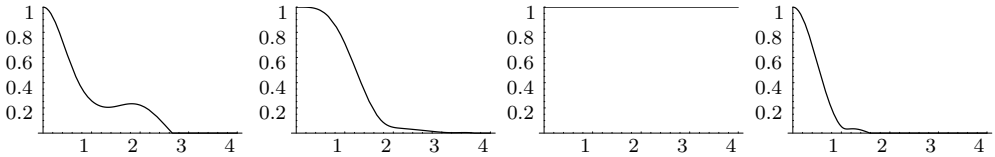

Figure 12. Concurrence versus time with the same environment interaction (one plot for each Bell state).

accordance with our expectations. For the first plot we chose two different random matrices, with the same coupling constant of $\lambda_1 = \lambda_2 = 0.5$. We can see that the concurrence quickly goes to and remains zero. For the second plot we decided to actually switch one of the qubit interactions off by setting $\lambda_1 = 0$ and keeping the other constant at 0.5. It can be seen that the qubits still fully decohere and all entanglement is lost. However note that the time taken is longer.

For the case of the same interaction for both qubits (9) we have figure 12. There are four plots, each corresponding to a different initial Bell state. Three of the setups tend to zero with some interesting irregularities along the way. However, one setup, corresponding to the Bell state: $(|01\rangle - \langle 10|)/\sqrt{2}$, exhibits no decoherence and entanglement is maintained at its maximal level. This behaviour is due to symmetry reasons. As the Hamiltonian is invariant under permutation of the qubits and as there is only one antisymmetric two-qubit state, such a state will decouple from the environment and not evolve in time. This is somehow the most elementary occurrence of a decoherence free subspace, see [11].

Acknowledgments

This work originated in a bilateral programme between the universities of KwaZulu-Natal (SA) and Leuven (BE) and was partially funded by a cooperation grant (BIL04/48, GUN: 2068811) between Flanders and South Africa. The authors hereby express their gratitude for the warm hospitality extended by the host institutes on the occasion of several exchanges.

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